SHADOWS OF 3-UNIFORM HYPERGRAPHS UNDER A MINIMUM DEGREE CONDITION

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Abstract. We prove a minimum degree version of the Kruskal–Katona theorem for triple systems: given $d \geq 1/4$ and a triple system $\mathcal{F}$ on n vertices with minimum degree $\delta(\mathcal{F}) \geq d\binom{n}{2}$, we obtain asymptotically tight lower bounds for the size of its shadow. Equivalently, for $t \geq n/2 - 1$, we asymptotically determine the minimum size of a graph on n vertices, in which every vertex is contained in at least $\binom{t}{2}$ triangles. This can be viewed as a variant of the Rademacher–Turán problem.

1. Introduction

Given a set X and a family $\mathcal{F}$ of k-subsets of X, the *shadow* $\partial\mathcal{F}$ of $\mathcal{F}$ is the family of all $(k-1)$-subsets of X contained in some member of $\mathcal{F}$. The Kruskal–Katona theorem [12, 13] is one of the most important results in extremal set theory – it gives a tight lower bound for the size of shadows of all k-uniform families of a given size. The following is a version due to Lovász [17]. Note that it is tight when t is an integer by considering the family of all k-subsets of a set of t vertices.

Theorem 1 (Kruskal–Katona theorem). *If $\mathcal{F}$ is a family of k-sets with $|\mathcal{F}| \geq \binom{t}{k}$ for some real number t, then $|\partial\mathcal{F}| \geq \binom{t}{k-1}$.*

A family $\mathcal{F}$ of k-subsets of X is often regarded as a *k-uniform hypergraph*, or *k-graph* $(X, \mathcal{F})$ with X as the vertex set and $\mathcal{F}$ as the edge set. For every $x \in X$, define $\mathcal{F}_x = \{F \setminus x : x \in F$ and $F \in \mathcal{F}\}$. The minimum (vertex) degree of $\mathcal{F}$ is denoted by $\delta(\mathcal{F}) := \min_x |\mathcal{F}_x|$. The following minimum degree version of the Kruskal–Katona theorem has not been studied before but emerged naturally when Han, Zang, and Zhao [9] investigated a packing problem for 3-graphs.

Problem 2. *Given $k \geq 3$ and $0 < d < 1$, let X be a set of n vertices and $\mathcal{F}$ be a family of k-subsets of X with $\delta(\mathcal{F}) \geq d\binom{n}{k-1}$.*[1] *How small can $|\partial\mathcal{F}|$ be?*

Problem 2 belongs to an area of active research on extremal problems under maximum or minimum degree conditions. Two early examples are the work of Bollobás, Daykin, and Erdős [1], who studied the minimum degree version of the Erdős matching conjecture, and of Frankl [6], who studied the Erdős–Ko–Rado theorem under maximum degree conditions. More recent examples include the minimum (co)degree Turán's problems [15, 18], the minimum degree version of the Erdős–Ko–Rado theorem [8, 10, 14], and the minimum degree version of Hilton–Milner

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[1] It is more natural to assume $\delta(\mathcal{F}) \geq d\binom{n-1}{k-1}$ as $\binom{n-1}{k-1}$ is the largest possible degree. However, since we are mainly interested in the asymptotics of $|\partial\mathcal{F}|$, we choose the simpler looking condition $\delta(\mathcal{F}) \geq d\binom{n}{k-1}$.

theorem [7, 14]. Recently Jung and Katona [11] studied minimum $|\partial\mathcal{F}|/|\mathcal{F}|$ among all k-graphs $\mathcal{F}$ with maximum degree $\Delta(\mathcal{F}) \le d$.

Since $\delta(\mathcal{F}) \ge d\binom{n}{k-1}$ implies that $|\mathcal{F}| \ge d\binom{n}{k}$, we could apply Theorem 1 to $\mathcal{F}$ but will not obtain a tight bound for $|\partial\mathcal{F}|$. A better approach is applying Theorem 1 to $\mathcal{F}_x$ for each vertex x. Since $|\mathcal{F}_x| \ge d\binom{n}{k-1} \ge \binom{d^{\frac{1}{k-1}}n}{k-1}$, by Theorem 1, we have $|\partial\mathcal{F}_x| \ge \binom{d^{\frac{1}{k-1}}n}{k-2} \ge d^{\frac{k-2}{k-1}}\binom{n}{k-2} + O(n^{k-3})$. Consequently,

$$|\partial\mathcal{F}| = \sum_x \frac{|\partial\mathcal{F}_x|}{k-1} \ge \frac{n}{k-1} d^{\frac{k-2}{k-1}} \binom{n}{k-2} + O(n^{k-2}) \ge d^{\frac{k-2}{k-1}} \binom{n}{k-1} + O(n^{k-2}). \tag{1}$$

This bound is tight (up to the error term) when the first inequality in (1) is asymptotically an equality, which occurs when $\mathcal{F}_x$ is a clique of order $d^{\frac{1}{k-1}}n$ for every x. Thus, the bound in (1) is asymptotically tight when $\mathcal{F}$ consists of $d^{\frac{1}{1-k}}$ vertex-disjoint cliques of order $d^{\frac{1}{k-1}}n$, in particular, when $d = \ell^{1-k}$ for some $\ell \in \mathbb{N}$.

In this paper we improve (1) and answer Problem 2 asymptotically for $k = 3$ and $d \ge 1/4$. Two overlapping cliques of order about $\sqrt{d}n + 1$ is a natural candidate for extremal hypergraphs – the following theorem confirms this for $\frac{1}{4} \le d < \frac{47-5\sqrt{57}}{24} \approx 0.385$. However, there is a different extremal hypergraph for larger values of d.

Theorem 3. *Let $1/4 \le d < 1$ and $n \in \mathbb{N}$ be sufficiently large. If $\mathcal{F}$ is a triple system on n vertices with $\delta(\mathcal{F}) \ge d\binom{n}{2}$, then*

$$|\partial\mathcal{F}| \ge \begin{cases} \left(4\sqrt{d} - 2d - 1\right)\binom{n}{2} & \text{if } \frac{1}{4} \le d < \frac{47-5\sqrt{57}}{24} \\ \left(\frac{1}{2} + \sqrt{\frac{4d-1}{12}}\right)\binom{n}{2} & \text{if } d \ge \frac{47-5\sqrt{57}}{24}. \end{cases}$$

These bounds are best possible up to an additive term of $O(n)$.

Although seemingly technical, Theorem 3 has an interesting application on *3-graph packing and covering.* Given positive integers a, b, c, let $K^3_{a,b,c}$ denote the complete 3-partite *3-graph* with parts of size a, b, and c. Answering a question of Mycroft [19], Han, Zang, and Zhao [9] determined the minimum $\delta(H)$ of a 3-graph H that forces a *perfect $K^3_{a,b,c}$-packing* in H for any given a, b, c.[2] One of the main steps in their proof is determining the smallest $\delta(H)$ of a 3-graph H that guarantees that every vertex of H is covered by a copy of $K^3_{a,b,c}$ (this is necessary for H containing a perfect $K^3_{a,b,c}$-packing).

Corollary 4. [9, Lemma 3.7] *Let $d_0 = 6 - 4\sqrt{2} \approx 0.343$. For any $\gamma > 0$, there exists $\eta > 0$ such that the following holds for sufficiently large n. If H is an n-vertex 3-graph with $\delta(H) \ge (d_0 + \gamma)\binom{n}{2}$, then each vertex of H is contained in at least $\eta n^{a+b+c-1}$ copies of $K^3_{a,b,c}$.*

It was shown [9, Construction 2.6] that d_0 in Corollary 4 is best possible. A proof of Corollary 4 can be found in [9] – we give a proof outline at the end of Section 2.

Our approach towards Theorem 3 is viewing it as an extremal problem on graphs. The following is an equivalent form of Problem 2, in which K^k_t denotes the complete k-graph on t vertices (and we omit the superscript when $k = 2$).

Problem 5. *Given a $(k-1)$-graph G on n vertices such that every vertex is contained in at least $d\binom{n}{k-1}$ copies of K^{k-1}_k, how many edges must G have?*

[2] Given hypergraphs H and F, a perfect F-packing in H is a spanning subgraph of H that consists of vertex-disjoint copies of F.

To see why Problems 2 and 5 are equivalent, let m_1 be the minimum $|\partial\mathcal{F}|$ for Problem 2 and m_2 be the minimum $e(G)$ for Problem 5. To see why $m_1 \geq m_2$, consider a k-uniform family $\mathcal{F}$ with $\delta(\mathcal{F}) \geq d\binom{n}{k-1}$. Let $G = (V(\mathcal{F}), \partial\mathcal{F})$ be the $(k-1)$-graph of its shadow. Since every member of $\mathcal{F}$ gives rise to a copy of K_k^{k-1} in G, $\delta(\mathcal{F}) \geq d\binom{n}{k-1}$ implies that every vertex is contained in at least $d\binom{n}{k-1}$ copies of K_k^{k-1}. Thus $|\partial\mathcal{F}| = e(G) \geq m_2$. To see why $m_2 \geq m_1$, consider a $(k-1)$-graph G such that every vertex is contained in at least $d\binom{n}{k-1}$ copies of K_k^{k-1}. Let $\mathcal{F}$ be the family of k-subsets of $V(G)$ that span copies of K_k^{k-1} in G. Then $\partial F \subseteq G$ and for every $v \in V(G)$, we have $|\mathcal{F}_v| \geq d\binom{n}{k-1}$. Thus $e(G) \geq |\partial\mathcal{F}| \geq m_1$ as desired.

In order to prove Theorem 3, we solve the $k = 3$ case of Problem 5 with $d \geq 1/4$. For convenience, we assume that every vertex of G is contained in at least $\binom{t}{2}$ triangles. There are essentially two *extremal graphs*: the first one consists of two copies of K_{t+1} that share $2t+2-n$ vertices; the second one is obtained from two disjoint copies of $K_{n/2}$ by adding a regular bipartite graph between them. The size of these two extremal graphs can be conveniently represented by a quadratic function $f(x)$, which arises naturally from a lower bound for $e(G)$ in Proposition 7.

Theorem 6. *Let $n \in \mathbb{N}$, $t, r \in \mathbb{R}$ such that $n/2 \leq t+1 \leq n$, $r \geq 0$, and*

$$\binom{\frac{n}{2}-1}{2} + 3\binom{r}{2} = \binom{t}{2}. \tag{2}$$

Define a function $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \binom{t}{2} + x(n-x) - \binom{n-x-1}{2}. \tag{3}$$

If G is an n-vertex graph such that each vertex is contained in at least $\binom{t}{2}$ triangles, then

$$e(G) \geq \begin{cases} f(t) & \text{if } r+t \leq \frac{5n}{6} \text{ or approximately } t \leq 0.6208n, \\ f(\frac{n}{2}+r-1) & \text{otherwise.} \end{cases} \tag{4}$$

Furthermore, these bounds are tight when $n/2, t, r$ are integers, and tight up to an additive $O(n)$ in general.

Theorem 6 can be viewed as a variant of the well-studied Rademacher–Turán problem. Starting with the work of Rademacher (unpublished) and of Erdős [4], the Rademacher–Turán problem studies the minimum number of triangles in a graph with given order and size. Instead of the total number of triangles in a graph, one may ask for the maximum or minimum number of triangles containing a fixed vertex. Given a graph G, we define the *triangle-degree* of a vertex as the number of triangles that contain this vertex. Let $\Delta_{K_3}(G)$ and $\delta_{K_3}(G)$ denote the maximum and minimum triangle-degree in G, respectively. The contrapositive of Theorem 6 states that *if G is a graph on n vertices that fails* (4)*, then $\delta_{K_3}(G) < \binom{t}{2}$*. Correspondingly, the *maximum* triangle-degree version of Rademacher–Turán problem was recently studied by Falgas-Ravry, Markström, and Zhao [5]. In addition, Theorem 6 looks similar to the question of Erdős and Rothschild [3] on the *book size* of graphs: in the complementary form, it asks for the maximum size of a graph on n vertices, in which every edge is contained in at most d triangles.

We prove Theorem 6 and Theorem 3 in the next section. When $t < n/2-1$, it is reasonable to speculate that an extremal graph is a disjoint union of copies of K_{t+1} and an extremal graph for Theorem 6. Unfortunately we cannot verify this. We provide some evidence for this speculation in the last section.

Notation. Given a family $\mathcal{F}$ of sets, $|\mathcal{F}|$ is the size of $\mathcal{F}$, namely, the number of sets in $\mathcal{F}$. A k-uniform hypergraph H, or k-graph, consists of a vertex set $V(H)$ and an edge set $E(H)$, which is a family of k-subsets of $V(H)$. Given a vertex set S, denote by $e_H(S)$ the number of edges of H induced on S. Suppose G is a graph. For a vertex $v \in V(G)$, let $N_G(v)$ denote the *neighborhood* of v, the set of vertices adjacent to v, and let $d_G(v) = |N_G(v)|$ be the degree of v. Let $N_G[v] := N_G(v) \cup \{v\}$ denote the *closed neighborhood* of v. When the underlying (hyper)graph is clear from the context, we omit the subscript in these notations.

2. Proofs of Theorem 6 and Thereom 3

Suppose that $G = (V, E)$ is a graph on n vertices such that each vertex is contained in at least $\binom{t}{2}$ triangles, in other words,

$$\forall v \in V, \quad e(N(v)) \geq \binom{t}{2}, \tag{5}$$

where t is a positive real number. Trivially $t \leq \delta(G) \leq n-1$ because $e(N(v)) \leq \binom{d(v)}{2}$ for every vertex $v \in V$. Therefore

$$e(G) \geq \frac{\delta(G)n}{2} \geq \frac{tn}{2}.$$

When $t+1$ divides n, this bound is tight because G can be a disjoint union of $\frac{n}{t+1}$ copies of K_{t+1}. Below we often assume that $t \leq n-2$ because when $t = n-1$, we must have $G = K_n$.

Let us derive another lower bound for $e(G)$ by using the function f defined in (3).

Proposition 7. *If $G = (V, E)$ is a graph on n vertices satisfying (5), then $e(G) \geq f(\delta(G))$, and the equality holds if and only if there exists $v_0 \in V$ such that $e(N(v_0)) = \binom{t}{2}$, $d(v) = \delta(G)$ for all $v \notin N(v_0)$, and $V \setminus N[v_0]$ induces a clique.*

Proof. Suppose $\delta(G) = \delta$ and $v_0 \in V$ satisfies $d(v_0) = \delta$. Since we may partition $E(G)$ into the edges induced on $N(v_0)$ and the edges incident to some vertex $v \notin N(v_0)$, we have

$$e(G) = e(N(v_0)) + \left(\sum_{v \notin N(v_0)} d(v) \right) - e(V \setminus N(v_0)).$$

Because of (5), $d(v) \geq \delta$ for all $v \notin N(v_0)$, and $e(V \setminus N(v_0)) \leq \binom{n-\delta-1}{2}$ (note that v_0 has no neighbor outside $N(v_0)$), we derive that $e(G) \geq \binom{t}{2} + \delta(n-\delta) - \binom{n-\delta-1}{2}$, and equality holds exactly when $e(N(v_0)) = \binom{t}{2}$, $d(v) = \delta(G)$ for all $v \notin N(v_0)$, and $V \setminus N[v_0]$ induces a clique. □

Let us construct three graphs satisfying (5). Note that, if r satisfies (2), then $r \leq n/2$ because $\binom{n/2-1}{2} + 3\binom{n/2}{2} = \binom{n-1}{2} \geq \binom{t}{2}$.

Construction 8. *Suppose $t, r \in \mathbb{R}$ satisfy $\frac{n}{2} - 1 \leq t \leq n-2$, $r \geq 0$, and (2).*

1. *Let G_1 be the union of two copies of $K_{\lceil t \rceil + 1}$ sharing $2\lceil t \rceil + 2 - n$ vertices.*
2. *When n is even, let G_2 be the n-vertex graph obtained from two disjoint copies of $K_{n/2}$ by adding an $\lceil r \rceil$-regular bipartite graph between two cliques.*
3. *When n is odd, let $r' \in \mathbb{R}^+$ satisfy $\binom{\frac{n-3}{2}}{2} + 3\binom{r'}{2} = \binom{t}{2}$. Let G_2' be the n-vertex graph obtained from two disjoint copies of $K_{(n-1)/2}$ by adding an $\lceil r' \rceil$-regular bipartite graph between them, and a new vertex whose adjacency is the exactly the same as one of the existing vertices.*

It is easy to see that G_1, G_2, G_2' all satisfy (5). For example, consider a vertex $x \in V(G_2)$. Let A and B denote the vertex sets of the two copies of $K_{n/2}$ of G_2 and assume $x \in A$. Then $N(x)$ contains $\binom{n/2-1}{2}$ edges from A, $\binom{\lceil r \rceil}{2}$ edges from B, and $\lceil r \rceil(\lceil r \rceil - 1)$ edges between A and B. Hence $e(N(x)) = \binom{\frac{n}{2}-1}{2} + 3\binom{\lceil r \rceil}{2} \geq \binom{t}{2}$.

The following proposition gives the sizes of G_1, G_2, and G_2'.

Proposition 9. *Suppose $n \in \mathbb{N}$, $t, r \geq 0$ satisfy $\frac{n}{2} - 1 \leq t \leq n-1$ and (2). If all $n/2, t, r$ are integers, then $e(G_1) = f(t)$ and $e(G_2) = f(n/2+r-1)$, otherwise $e(G_1) \leq f(t) + n$ and $e(G_2) \leq f(n/2+r-1) + n/2$. Furthermore, $e(G_2') = f(n/2+r-1) + O(n)$ when $r', r = \Omega(n)$.*

Proof. First, by the definition of $f(x)$, it is easy to see that

$$f(t) = \binom{n}{2} - (n-1-t)^2 \tag{6}$$

(alternatively when $t \in \mathbb{Z}$, we can apply Proposition 7 by letting v_0 be any vertex not in the intersection of the two cliques). We know that

$$e(G_1) = \binom{n}{2} - (n-1-\lceil t \rceil)^2 \geq \binom{n}{2} - (n-1-t)^2 = f(t)$$

and equality holds when $t \in \mathbb{Z}$. In addition, we have $e(G_1) \leq f(t) + n$ because

$$(n-1-\lceil t \rceil)^2 - (n-1-t)^2 = (2(n-1) - (\lceil t \rceil + t))(\lceil t \rceil - t) \leq n$$

by using $t+1 \geq \lceil t \rceil \geq t \geq n/2 - 1$.

Second, using the definitions of $f(x)$ and r, it is not hard to see that

$$f\left(\frac{n}{2} + r - 1\right) = \frac{n}{2}\left(\frac{n}{2} + r - 1\right). \tag{7}$$

It follows that

$$e(G_2) = \frac{n}{2}\left(\frac{n}{2} + \lceil r \rceil - 1\right) \leq f\left(\frac{n}{2} + r - 1\right) + \frac{n}{2}$$

and equality holds when $r \in \mathbb{Z}$.

Third, it is easy to see that

$$e(G_2') = \frac{n+1}{2}\left(\frac{n-1}{2} + \lceil r' \rceil - 1\right).$$

By the definitions of r and r', we have $\binom{r'}{2} - \binom{r}{2} = \frac{2n-7}{24}$. When $r, r' = \Omega(n)$, we have $r' - r = O(1)$ and consequently,

$$\begin{aligned} e(G_2') - f\left(\frac{n}{2} + r - 1\right) &\leq \frac{n+1}{2}\left(\frac{n-1}{2} + r' - 1\right) - \frac{n}{2}\left(\frac{n}{2} + r - 1\right) \\ &= \frac{n}{2}(r' - r) + \frac{r'}{2} - \frac{3}{4} = O(n). \end{aligned}$$

□

We compare $f(t)$, the approximate size of G_1, with $f(\frac{n}{2} + r - 1)$, the approximate size of G_2 and G_2', in the next proposition.

Proposition 10. *Suppose $\frac{n}{2} - 1 \leq t \leq n-1$, $f(x)$ and r are defined in (3) and (2), respectively. We have $f(t) \leq f(\frac{n}{2} + r - 1)$ if and only if $r + t \leq \frac{5n}{6}$, equivalently,*

$$t \leq \frac{5}{4}n - \frac{\sqrt{57n^2 - 72n}}{12} - 1 \approx 0.6208n. \tag{8}$$

To prove Proposition 10, we need a simple fact on quadratic functions.

Fact 11. *Suppose $g(x)$ is a quadratic function with a maximum at $x = a$ and assume $x_1 \leq x_2$. Then $g(x_1) \leq g(x_2)$ if and only if $x_1 + x_2 \leq 2a$.* □

Proof of Proposition 10. First note that

$$f(x) = -\frac{3}{2}x^2 + \frac{4n-3}{2}x - \frac{n^2}{2} + \binom{t}{2} + \frac{3}{2}n - 1$$

is a quadratic function with a maximum at $x = \frac{2n}{3} - \frac{1}{2}$. Second, since $r \le \frac{n}{2}$, it follows that

$$\binom{\frac{n}{2}+r-1}{2} = \binom{\frac{n}{2}-1}{2} + \left(\frac{n}{2}-1\right)r + \binom{r}{2} \ge \binom{\frac{n}{2}-1}{2} + 3\binom{r}{2} = \binom{t}{2}.$$

Consequently $\frac{n}{2}+r-1 \ge t$. By Fact 11, $f(t) \le f(\frac{n}{2}+r-1)$ if and only if $t + \frac{n}{2} + r - 1 \le \frac{4n}{3} - 1$ or $r + t \le \frac{5n}{6}$. By (2), this is equivalent to

$$\binom{\frac{n}{2}-1}{2} + 3\binom{\frac{5n}{6}-t}{2} \ge \binom{t}{2} \quad \text{or} \quad (t+1)^2 - \frac{5}{2}(t+1)n + \frac{7}{6}n^2 + \frac{n}{2} \ge 0,$$

which holds exactly when $t+1 \le \frac{5}{4}n - \frac{\sqrt{57n^2-72n}}{12}$ (because $t < n$). □

We are ready to prove Theorem 6.

Proof of Theorem 6. Assume that $\delta = \delta(G)$. We separate two cases.

Case 1: $r + t \le \frac{5n}{6}$, equivalently, (8).

First assume that $\delta \ge \frac{4}{3}n - t - 1$. Since $t \le \frac{5n}{6} - r$, we have $\delta \ge \frac{n}{2} + r - 1$ and consequently,

$$e(G) \ge \frac{n}{2}\left(\frac{n}{2}+r-1\right) = f\left(\frac{n}{2}+r-1\right) \ge f(t)$$

by (7) and Proposition 10.

Second assume that $\delta < \frac{4}{3}n - t - 1$. By Proposition 7, we have $e(G) \ge f(\delta)$. Recall that (5) forces $t \le \delta$. Since $t \le \delta < \frac{4}{3}n - t - 1$ and $f(x)$ is a quadratic function maximized at $\frac{2n}{3} - \frac{1}{2}$, we derive from Fact 11 that $f(\delta) \ge f(t)$. Hence $e(G) \ge f(\delta) \ge f(t)$.

Case 2: $r + t > \frac{5n}{6}$.

If $\delta \ge \frac{n}{2} + r - 1$, then $e(G) \ge \frac{n}{2}(\frac{n}{2}+r-1) = f(\frac{n}{2}+r-1)$ by (7). Otherwise $\delta < \frac{n}{2} + r - 1$. Note that

$$\delta + \frac{n}{2} + r - 1 \ge t + \frac{n}{2} + r - 1 > \frac{5n}{6} + \frac{n}{2} - 1 = \frac{4n}{3} - 1.$$

Since the quadratic function $f(x)$ is maximized at $\frac{2n}{3} - \frac{1}{2}$, we derive from Fact 11 that $f(\delta) \ge f(\frac{n}{2}+r-1)$. By Proposition 7, we have $e(G) \ge f(\delta) \ge f(\frac{n}{2}+r-1)$.

By Proposition 9, when $n/2, t, r$ are all integers, we have $e(G_1) = f(t)$ and $e(G_2) = f(\frac{n}{2}+r-1)$. In other cases, we have $e(G_1) \le f(t) + n$ and $e(G_2) \le f(\frac{n}{2}+r-1) + n/2$. When n is odd and $r + t > 5n/6$, we have $r, r' = \Omega(n)$ and thus $e(G_2') = f(\frac{n}{2}+r-1) + O(n)$. □

Remark 12. *When $n/2, t, r$ are all integers, we actually learn the following about extremal graphs from the proof of Theorem 6. Suppose that G is an extremal graph. We claim that $G = G_1$ when $r + t < 5n/6$, and G is $(n/2+r-1)$-regular when $r + t > 5n/6$,.*

Indeed, first assume $r + t < 5n/6$. If $\delta \ge \frac{4}{3}n - t - 1$, then $\delta > \frac{n}{2} + r - 1$ and consequently, $e(G) > \frac{n}{2}(\frac{n}{2}+r-1) = f(t)$, a contradiction. Following the second case of Case 1, we obtain that $e(G) = f(\delta) = f(t)$ and consequently, $\delta = t$. Using Proposition 7, we can derive that $G = G_1$. When $r + t > 5n/6$, the second case of Case 2 shows that $e(G) \ge f(\delta) > f(\frac{n}{2}+r-1)$, a contradiction. Thus $\delta \ge \frac{n}{2}+r-1$ and $e(G) = \frac{n}{2}(\frac{n}{2}+r-1)$, which forces G to be $(n/2+r-1)$-regular.

We now prove Theorem 3 by applying Theorem 6 and the arguments that show the equivalence of Problems 2 and 5 in Section 1.

Proof of Theorem 3. Suppose $1/4 \leq d < 1$ and $n \in \mathbb{N}$ is sufficiently large. Choose $t \in \mathbb{R}^+$ such that $\binom{t}{2} = d\binom{n}{2}$. Since $\binom{\sqrt{d}n}{2} < d\binom{n}{2} < \binom{\sqrt{d}n+1}{2}$, we have $\sqrt{d}n < t < \sqrt{d}n + 1$.

Suppose $\mathcal{F}$ is a triple system on n vertices with $\delta(\mathcal{F}) \geq d\binom{n}{2}$. Let $G = (V(\mathcal{F}), \partial\mathcal{F})$ be the graph whose edge set is the shadow $\partial\mathcal{F}$. For every $x \in V(G)$, we have $e_G(N(x)) \geq d\binom{n}{2}$.

Case 1: $\frac{1}{4} \leq d < \frac{47-5\sqrt{57}}{24}$.

Thus $\frac{1}{2} \leq \sqrt{d} < \frac{15-\sqrt{57}}{12}$. Since n is sufficiently large, we have $\sqrt{d}n \leq \frac{15-\sqrt{57}}{12}n - 2$. Since $\sqrt{d}n < t < \sqrt{d}n + 1$, it follows that

$$\frac{n}{2} < t < \frac{15-\sqrt{57}}{12}n - 1 < \frac{5}{4}n - \frac{\sqrt{57n^2 - 72n}}{12} - 1.$$

This allows us to apply the first case of Theorem 6 and (6) to derive that

$$\begin{aligned} e(G) \geq f(t) &= \binom{n}{2} - (n-1-t)^2 \geq \binom{n}{2} - (n-1-\sqrt{d}n)^2 \\ &= (4\sqrt{d} - 2d - 1)\binom{n}{2} + n - dn - 1 \\ &\geq (4\sqrt{d} - 2d - 1)\binom{n}{2} \quad \text{as } d < 1 \text{ and } n \text{ is sufficiently large.} \end{aligned}$$

Case 2: $d \geq \frac{47-5\sqrt{57}}{24}$.

Thus $\sqrt{d} \geq \frac{15-\sqrt{57}}{12}$. Since $t > \sqrt{d}n$, it follows that

$$t + 1 > \frac{15-\sqrt{57}}{12}n + 1 > \frac{5}{4}n - \frac{\sqrt{57n^2 - 72}}{12}$$

because $\sqrt{57n^2 - 72} > \sqrt{57n^2} - 6$ for $n \geq 2$. Since (8) fails, we will apply the second case of Theorem 6. Since $\binom{t}{2} = d\binom{n}{2}$ and $r \geq 0$, we can obtain from (2) that

$$r = \frac{1}{6}\left(3 + \sqrt{3(n-1)\big((4d-1)n+5\big)}\right) = \frac{1}{2} + \frac{n}{2}\sqrt{\frac{4d-1}{3}} + h(n),$$

where

$$h(n) = \frac{1}{2\sqrt{3}}\left(\sqrt{(4d-1)n^2 + (6-4d)n - 5} - \sqrt{4d-1}n\right).$$

It is easy to see that $0 \leq h(n) = O(1)$. Theorem 6 thus gives that

$$\begin{aligned} e(G) &\geq f\left(\frac{n}{2} + r - 1\right) = \frac{n}{2}\left(\frac{n}{2} + r - 1\right) \\ &= \frac{n}{2}\left(\frac{n}{2} - \frac{1}{2} + \frac{n}{2}\sqrt{\frac{4d-1}{3}} + h(n)\right) \\ &= \binom{n}{2}\left(\frac{1}{2} + \sqrt{\frac{4d-1}{12}}\right) + \frac{n}{4}\sqrt{\frac{4d-1}{3}} + \frac{n}{2}h(n) \qquad (9) \\ &\geq \binom{n}{2}\left(\frac{1}{2} + \sqrt{\frac{4d-1}{12}}\right). \end{aligned}$$

To see why these bounds are asymptotically tight, for every graph $G \in \{G_1, G_2, G_2'\}$, we construct a triple system $\mathcal{F}_G$ whose members are all triangles of G. Then $\partial\mathcal{F}_G \subseteq E(G)$ and $\delta(\mathcal{F}_G) \geq \binom{t}{2} = d\binom{n}{2}$.

Proposition 9 gives that $|\partial\mathcal{F}_{G_1}| \le e(G_1) \le f(t) + n$. By (6) and the assumption $t \le \sqrt{d}n + 1$,

$$
\begin{aligned}
|\partial\mathcal{F}_{G_1}| \le f(t) + n &\le \binom{n}{2} - (n - 2 - \sqrt{d}n)^2 + n \\
&= \left(4\sqrt{d} - 2d - 1\right)\binom{n}{2} + \left(3 - 2\sqrt{d} - d\right)n - 4 + n \\
&= \left(4\sqrt{d} - 2d - 1\right)\binom{n}{2} + O(n).
\end{aligned}
$$

When n is even, we apply Proposition 9 and (9) obtaining that

$$|\partial\mathcal{F}_{G_2}| \le e(G_2) \le f\left(\frac{n}{2} + r - 1\right) + \frac{n}{2} = \binom{n}{2}\left(\frac{1}{2} + \sqrt{\frac{4d-1}{12}}\right) + O(n).$$

When n is odd, we assume $r + t > 5n/6$ and thus $r, r' = \Omega(n)$. By Proposition 9 and (9), we conclude that

$$|\partial\mathcal{F}_{G_2'}| \le e(G_2') = f\left(\frac{n}{2} + r - 1\right) + O(n) = \binom{n}{2}\left(\frac{1}{2} + \sqrt{\frac{4d-1}{12}}\right) + O(n). \qquad \square$$

We outline the proof of Corollary 4 emphasizing how Theorem 3 is applied. In a 3-graph, the *degree* of a pair p of vertices is the number of the edges that contains p.

Proof Outline of Corollary 4. Assume $\eta \ll \gamma$ and $\varepsilon = \gamma/12$. Let H be an n-vertex 3-graph and x be a vertex of H. In order to find $\eta n^{a+b+c-1}$ copies of $K^3_{a,b,c}$, it suffices to find $\frac{\gamma}{2}\binom{n}{2}$ pairs of vertices of H_x with degree at lease $\varepsilon^2 n$ – this follows from standard counting arguments in extremal (hyper)graph theory, or conveniently [16, Lemma 4.2] of Lo and Markström.

Suppose $\delta_1(H) \ge (d_0 + \gamma)\binom{n}{2}$ with $d_0 = 6 - 4\sqrt{2} \approx 0.343$. As shown in [9, Lemma 3.3], it is easy to find a set V_0 of at most $3\varepsilon n$ vertices and a subgraph H' of H on $V \setminus V_0$ such that $\delta(H') \ge d_0\binom{n'}{2}$, where $n' = |V \setminus V_0|$, and every pair in $\partial H'$ has degree at least $\varepsilon^2 n$ in H. Since $\frac{1}{4} < d_0 < \frac{47-5\sqrt{57}}{24} \approx 0.385$, by the first case of Theorem 3, we have

$$|\partial H'| \ge (4\sqrt{d_0} - 2d_0 - 1)\binom{n'}{2} \ge \left(4\sqrt{d_0} - 2d_0 - 1 - \frac{\gamma}{2}\right)\binom{n}{2}.$$

For every vertex $x \in V(H)$, since $d(x) \ge (d_0 + \gamma)\binom{n}{2}$ and crucially $4\sqrt{d_0} - 2d_0 - 1 = 1 - d_0$, at least $\frac{\gamma}{2}\binom{n}{2}$ pairs in H_x are also in $\partial H'$ thus having degree at lease $\varepsilon^2 n$, as desired.

$\square$

3. Concluding remarks

Let us restate the $k = 3$ case of Problem 5.

Problem 13. *Let G be a graph on n vertices such that each vertex is contained in at least $\binom{t}{2}$ triangles, where t is a positive real number. How many edges must G have?*

Our Theorem 6 (asymptotically) answers Problem 13 for $n/2 \le t + 1 \le n$. The following proposition shows that for larger n, all but $O(t^3)$ vertices of an extremal graph are contained in isolated copies of K_{t+1}.

Proposition 14. *When $n > (t+1)^2(t+2)/4$, every extremal graph for Problem 13 contains an isolated copy of K_{t+1}.*

Proof. Let $G = (V, E)$ be an extremal graph with $|V| = n$. Since every vertex lies in at least $\binom{t}{2}$ triangles, it suffices to show that G contains a vertex of degree t and all of its neighbors also have degree t (thus inducing an isolated copy of K_{t+1}).

Suppose $n = a(t+1) + b$, where $0 < b \leq t$. Let G' be the disjoint union of $a-1$ copies of K_{t+1} together with two copies of K_{t+1} sharing $t+1-b$ vertices. Since G is extremal, we have

$$2e(G) \leq 2e(G') = tn + (t+1-b)b \leq tn + (t+1)^2/4.$$

Partition $V(G)$ into $A \cup B$ such that A consists of all vertices of degree greater than t and B consists of all vertices of degree exactly t. Then

$$\sum_{v\in A}(d_G(v) - t) = \sum_{v\in V}(d_G(v) - t) = 2e(G) - tn \leq (t+1)^2/4.$$

This implies that $|A| \leq (t+1)^2/4$. Let $e(A, B)$ denote the number of edges (of G) between A and B. It follows that

$$e(A, B) \leq \sum_{v\in A} d(v) \leq \frac{1}{4}(t+1)^2 + t|A| \leq \frac{1}{4}(t+1)^3.$$

Let B' consists of the vertices of B that are adjacent to some vertex of A. Then $|B'| \leq e(A, B) \leq (t+1)^3/4$. If $n > (t+1)^2(t+2)/4$, then $n > |A| + |B'|$ and consequently, there exists a vertex of B whose t neighbors are all in B, as desired. □

The $t = 2$ case of Problem 13 assumes that every vertex in an n-vertex graph is covered by a triangle. Since $\delta(G) \geq 2$, it follows that $e(G) \geq n$, which is best possible when 3 divides n. Recently, Chakraborti and Loh [2] determined the minimum number of edges an n-vertex graph in which every vertex is contained in a copy of K_s, for arbitrary $s \leq n$. Their extremal graph is the union of copies of K_s, all but two of which are isolated.

Finally, using careful case analysis, we can answer Problem 13 *exactly* when t is very close to n. This falls into the $r + t > 5n/6$ case of Theorem 6 but G_2 defined in Construction 8 is not necessarily extremal (unless both r and $n/2$ are integers).

- When $n = t + 2$, the (unique) extremal graph is K_n^-, the complete graph on n vertices minus one edge.
- When $n = t + 3$ is even, the (unique) extremal graph is K_n minus a perfect matching (provided $t > 5$). When $n = t+3$ is odd, K_n minus a matching of size $\frac{n-1}{2}$ is an extremal graph (provided $t > 6$).
- When $n = t + 4$, the complement of any K_3-free 2-regular graph on n vertices is an extremal graph. Note that $r = n/2 - 2$ in this case and thus G_2 is one of the extremal graphs when n is even.

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